

Network as a computer: ranking paths to find flows

Dusko Pavlovic*

Oxford University and Kestrel Institute

Abstract. We explore a simple mathematical model of network computation, based on Markov chains. Similar models apply to a broad range of computational phenomena, arising in networks of computers, as well as in genetic, and neural nets, in social networks, and so on. The main problem of interaction with such spontaneously evolving computational systems is that the data are not uniformly structured. An interesting approach is to try to extract the semantical content of the data from their distribution among the nodes. A concept is then identified by finding the community of nodes that share it. The task of data structuring is thus reduced to the task of finding the network communities, as groups of nodes that together perform some non-local data processing. Towards this goal, we extend the ranking methods from nodes to paths, which allows us to extract information about the likely flow biases from the available static information about the network.

1 Introduction

Initially, Web search was developed as an instance of *information retrieval*, optimized for a particularly large distributed database. With the advent of online advertising, Web search got enhanced by a broad range of *information supply* techniques where the search results are expanded by additional data, extrapolated from user's interests, and from search engine's stock of information. From the simple idea to match and coordinate the push and the pull of information on the Web as a new computational platform [18] sprang up a new generation of web businesses and social networks. Similar patterns of information processing are found in many other evolutionary systems, from gene regulation, protein interaction and neural nets, through the various networks of computers and devices, to the complex social and market structures [15].

This paper explores some simple mathematical consequences of the observation that the Web, and similar networks, are much more than mere information repositories: besides storing, and retrieving, and supplying information, they also generate, and process information. We pursue the idea that the Web can be modeled as a computer, rather than a database; or more precisely, as a vast multi-party computation [6], akin to a market place, where masses of selfish agents jointly evaluate and generate public information, driven by their private utilities. While this view raises interesting new problems across the whole gamut of Computer Science, the most effective solutions, so far, of the problem of *semantical* interactions with the Web computations were obtained by rediscovering and adopting the *ranking* methods, deeply rooted in the sociometric tradition [11,10], and adapting them for use on very large indices, leading to the whole new paradigm of *search* [19,12,13]. Implicitly, the idea of the Web as a computer is tacitly present already in this paradigm, in the sense that the search rankings are extracted from the link structure, and other intrinsic information, generated on the Web itself, rather than stored in it.

Outline of the paper. In section 2 we introduce the basic network model, and describe a first attempt to extract information about the flows through a network from the available static data about it. In sections 3 and 4, we describe the structure which allows us to lift the notion of rank, described in section 5, to path networks in section 6. Ranking paths allows us to extract a random variable, called attraction bias, which allows measuring the *mutual information* of the distributions of the inputs and the outputs of the network computation, which can be viewed as an indicator of non-local information processing that takes place in the given network. In the final section, we describe how the obtained data can be used to detect semantical

* Email: dusko@kestrel.edu,comlab.ox.ac.uk. Supported by EPSRC and ONR.

structures in a network. The experimental work necessary to test the practical effectiveness of the approach is left for future work.

2 Networks

Basic model. We view a network as an edge-labelled directed graph $A = (R \xleftarrow{\gamma} E \xrightarrow[\varrho]{\delta} N)$, where N and E are, respectively, the finite sets of *nodes*, and *links*, or *edges*, whereas R is an ordered field of *values* (in some applications an ordered ring of functions). A link $i \xrightarrow{e} j$ is thus an element $e \in E$ with $\delta(e) = i$ and $\varrho(e) = j$. The value $\gamma(e)$ is the cost (when positive), or payoff (when negative) of the traffic over e . These data induce the *adjacency matrix* $E = (E_{ij})_{N \times N}$ and the *capacity matrix* $A = (A_{ij})_{N \times N}$, with the entries $E_{ij} = \{e \in E \mid i \xrightarrow{e} j\}$ and $A_{ij} = \sum_{e \in E_{ij}} A_e$, where $A_e = 2^{-\gamma(e)}$ is the capacity of the link e .

Remark. The term "capacity" is used here as in network flow theory.¹ The cost or the payoff of a link may represent its value in a pay-per-click model of a fragment of the Web; or it may denote the proximity of the web pages within the same site, or within a group of interconnected sites. In a protein network, the energy cost or payoff may be derived from the chemical affinities between the nodes. While this parameter can be abstracted away, simply by taking $\gamma(e) = 0$ for all links e , its role will become clear in sections 3 and 6, where it allows discounting and eliminating some paths.

Basic dynamics. The simplest model of network dynamics is based on the assumption that the traffic flows are distributed proportionally to the link capacities. Randomly sampling the Web traffic, we shall thus find a surfer on a link e with the probability $\alpha_e = \frac{A_e}{A_\bullet}$, where $A_\bullet = \sum_{f \in E} A_f$.

In order to find the communities in a network, we need to detect the traffic biases between its nodes. We assume that the traffic between the nodes within the same community will be higher than the capacity of the links between them would lead us to expect; and that the traffic between the different communities will be lower than expected. To measure such traffic biases, we normalize the capacity matrix A to get the *capacity distribution* $\alpha = (\alpha_{ij})_{N \times N}$ as $\alpha_{ij} = \frac{A_{ij}}{A_{\bullet\bullet}}$, where $A_{\bullet\bullet} = \sum_{k, \ell \in N} A_{k\ell}$. The entry α_{ij} is thus the probability that a random sample of traffic on A , following the simple dynamics proportional to capacity, will be found on a link from i to j . On the other hand, the marginals of the probability distribution α ,

$$\alpha_{i\bullet} = \sum_{j \in N} \alpha_{ij} \quad \alpha_{\bullet j} = \sum_{i \in N} \alpha_{ij}$$

correspond, respectively, to the probabilities that a random sample of traffic will have i as its source, and j as its destination. Let us call $\alpha_{i\bullet}$ the *out-rank* of i , and $\alpha_{\bullet j}$ the *in-rank* of j , because they can be viewed as the simplest, albeit degenerate cases of the notion of rank.

If the in-rank and the out-rank are statistically independent, then (by the definition of independence) the probability that a random traffic sample goes from i to j will be $\alpha_{i\bullet}\alpha_{\bullet j}$. Their dependency is thus measured by the *traffic bias* matrix $v = (v_{ij})_{N \times N}$ with the entries $v_{ij} = \alpha_{ij} - \alpha_{i\bullet}\alpha_{\bullet j}$ falling in the interval $[-1, 1]$. The higher the bias, the more unexpected traffic there is. For a set of nodes $U \subseteq N$ the values

$$\text{Coh}(U) = \sum_{i, j \in U} v_{ij} \quad \text{Adh}(U) = \sum_{i \in U, j \notin U} v_{ij} + v_{ji}$$

can thus be construed as the *cohesion* and the *adhesion* forces: the total traffic bias within the group, and with its exterior, respectively. A network community U can thus be recognized as a set of nodes with a high cohesion and a low adhesion [16]. The idea that semantically related nodes can be captured as members

¹ The information theoretic homonym has a different, albeit related meaning, which motivates the choice of $\gamma(e) = -\log_2 A_e$.

of the same network communities, derived from the graph structure, is a natural extension of the ranking approach, which has been formalized in [9,17].

The only problem with applying that idea to the above model is that our initial assumption — that the traffic distribution on A is proportional to its link capacities — is not very realistic. It abstracts away all traffic dynamics. On the other hand, the static network model, as given above, does not provide any data about the actual traffic. We explore the ways to solve this problem, and extract increasingly more realistic views of traffic dynamics from a static network model.

3 Adding paths

A path $i \xrightarrow{a} j$ in a network A is a sequence of links $i \xrightarrow{a_1} k_1 \xrightarrow{a_2} k_2 \rightarrow \dots \xrightarrow{a_n} j$. In many cases of interest, traffic dynamics on a network depends on the path selections, rather than just on single links.

One idea is to add the paths to the structure of a network, and to annotate how the links compose into paths, and how the paths compose into longer paths. This amounts to generating the free category [14] over the network graph. Unfortunately, adding all paths to a network usually destroys some essential information, just like the transitive closure of a relation does. E.g., in a social network, a friend of a friend is often not even an acquaintance. Taking the transitive closure of the friendship relation obliterates that fact. Moreover, the popular "small world" phenomenon suggests that almost *every two people* can be related through no more than six friends of friends of friends... So already adding all paths of length six to a social network, with a symmetric friendship relation, is likely to generate a complete graph. In fact, the average probability that two of node's neighbors in an undirected graph are also linked with each other is an important factor, called *clustering coefficient* [20]. On the other hand, in some networks, e.g. of protein interactions, a link $i \rightarrow k$ which shortcuts the links $i \rightarrow j \rightarrow k$ often denotes a direct *feed-forward* connection, rather than a composition of the two links, and leads to essentially different dynamics.

So only "short" paths must be added to a network: composition must be penalized.

Definition 1. For a given network $A = (R \xleftarrow{\gamma} E \xrightarrow[\varrho]{\delta} N)$, a cutoff value $v \in R$, and a composition

penalty $d \in R$, we define the v -completion to be the network $A^{*v} = (R \xleftarrow{\gamma} E^{*v} \xrightarrow[\varrho]{\delta} N)$, where

$E^{*v} = \{a \in E^* \mid \gamma(a) \leq v\}$ and

$$\gamma(i_0 \xrightarrow{a_1} i_1 \xrightarrow{a_2} i_2 \rightarrow \dots \xrightarrow{a_n} i_n) = (n-1)d + \gamma(a_1) + \dots + \gamma(a_n)$$

and E^* is the set of all nonempty paths in A .

Remarks. E^* can be obtained as the matrix of sets $E^* = \sum_{n=0}^{\infty} E^n$ where each E^n is a power of the adjacency matrix E . If the entry E_{ij} is viewed as the set of links $\{i \xrightarrow{e} j\}$, then the entry $E_{ij}^2 = \sum_{k=1}^N E_{ik} \cdot E_{kj}$ corresponds to the set of 2-hop paths $\{i \xrightarrow{e_1} k \xrightarrow{e_2} j\}$ through the various nodes k ; the matrix E^3 similarly corresponds to the matrix of 3-hop paths, and so on.

The v -closed network A^{*v} is closed under the composition of low cost paths, but not if the cost is greater than v . It is not hard to see that the v -completion is an idempotent operation, i.e. $A^{*v*v} = A^{*v}$, but it may fail to be a proper closure operation, because a link e in A , with $\gamma(e) > v$, may lead to $A \not\subseteq A^{*v}$.

In the rest of the paper, we assume that the networks are v -complete for some v , i.e. $A = A^{*v}$. This means that the relevant pathways are already represented as links, with the composition penalty absorbed in the cost.

4 Network dynamics

In order to derive network dynamics from a static network model, one first specifies the way in which the behavior of a computational agent, processing data on the network, is influenced by the network structure, and then usually derives a Markov chain that drives the traffic. The network features that influence its dynamics can then be incrementally refined, yielding more and more information.

4.1 Forward and backward

Random walks on networks are often represented in terms of the behavior of surfers on the Web, following the hyperlinks.² The simplest surfer behavior chooses an out-link uniformly at random at each node. A visitor of a node i will thus proceed to a node j with probability $A_{ij}^> = \frac{A_{ij}}{A_{i\bullet}}$, where $A_{i\bullet} = \sum_{k=1}^N A_{ik}$ is the out-degree of i . The row-stochastic matrix $A^> = (A_{ij}^>)_{N \times N}$ represents *forward dynamics* of a network A . The entries $A_{ij}^>$ are called the *pull* coefficients of i by j .

Dually, *backward dynamics* of a network A is represented by a column-stochastic matrix $A^< = (A_{ij}^<)_{N \times N}$, where the entry $A_{ij}^< = \frac{A_{ij}}{A_{\bullet j}}$, with $A_{\bullet j} = \sum_{k=1}^N A_{kj}$ denoting the in-degree of j , describes the probability that a surfer who is on the node j came there from the node i . The entries $A_{ij}^<$ are called the *push* coefficients.

Remark. Note that the capacity matrix can be normalized to get $A^>$ and $A^<$ as above only if no rows, resp. columns, consist of 0s alone. This means that every node of the network must have at least one out-link, resp. at least one in-link. Networks that do not satisfy this requirement need to be modified, in one way or another, in order to enable analysis. Adding a high-cost link between every two nodes is clearly the minimal perturbation (with maximal entropy) that achieves this. Alternatively, the problem can also be resolved by adjoining a fresh node, and the high-cost links in and out of it [2]. Either way, the quantitative effect of such modifications can be made arbitrarily small by increasing the cost of the added links.

4.2 Forward-out and backward-in dynamics

The next example can be interpreted in two ways, either to show how forward and backward dynamics can be refined to take into account various navigation capabilities, or how to abstract away irrelevant cycles. Suppose that a surfer searches for the hubs on the network: he prefers to follow the hyperlinks that lead to the nodes with a higher out-degree. This preference may be realized by annotating the hyperlinks according to the out-rank of their target nodes. Alternatively, the surfer may explore the hyperlinks ahead, and select those with the highest out-degree; but we want to ignore the exploration part, and simply assume that he proceeds according to the out-rank of the nodes ahead. The probability that this surfer will move from i to j is thus

$$A_{ij}^{\triangleright} = A_{ij}^> \cdot \alpha_{j\bullet} = \frac{A_{ij} A_{j\bullet}}{A_{i\bullet} A_{\bullet\bullet}}$$

We call this the *forward-out* dynamics. In the dual, *backward-in* dynamics, the surfers are more likely to arrive to j from i if this is a frequently visited node, i.e. if its in-rank is higher

$$A_{ij}^{\triangleleft} = \alpha_{\bullet i} \cdot A_{ij}^< = \frac{A_{\bullet i} A_{ij}}{A_{\bullet\bullet} A_{\bullet j}}$$

These dynamics will be the particularly convenient to demonstrate an example of bias analysis in section 6, because they clearly display clearly how the simple traffic bias v from section 2 can be refined by the various dynamics factors.

4.3 Teleportation and preference

The main point of formulating network dynamics, especially in the Markov chain form, is to be able to compute the node ranking as its invariant distribution. However, since the network graphs are usually *not* strongly connected, the Markov chains, derived from their structure, are often reducible to classes of nodes with no way out.

The simplest remedy is the idea of *teleportation*, going back to [19]. A general interpretation is that, whichever dynamics a surfer might follow, at each node he tosses a biased coin, and with a probability $d \in (0, 1)$ follows that dynamics. Otherwise, with a probability $1 - d$, he "teleports" to a randomly chosen

² The surfers deserve their name by following the "waves", i.e. obeying the same dynamics.

node, ignoring all hyperlinks and other structure. Following, say, forward dynamics, the probability that he will go from i to j is thus $A_{ij}^P = dA_{ij}^\triangleright + \frac{1-d}{N}$. This is roughly the PageRank dynamics, from which the Google search engine had started [19]³. The induced dynamics is thus $A^P = dA^\triangleright + (1-d)P$, where $P = (P_{ij})_{N \times N}$ has all entries $P_{ij} = \frac{1}{N}$. In the networks without a cost function, this is interpreted as adding a link between every two nodes. The influence of such links can be controlled using the cost functions. In any case, the resulting Markov chains become irreducible, and their stationary distributions do not get captured in any closed components. Furthermore, the model can be *personalized* by capturing surfer's preferences in terms of the biases in P : the entries P_{ij} can be interpreted as i 's *trust* in j [8]. The extensions of the *backward* dynamics by teleportation yields to different interpretations, which the reader may wish to consider on her own.

5 Ranking

Intuitively, the rank of a node is the probability that randomly sampled traffic will be found to visit that node. In search, this is taken as a generic relevance measure. The technical implication is that the rank can be obtained as a stationary distribution of the Markov chain capturing dynamics. Each notion of dynamics thus induces a corresponding notion of rank. Since a Markov chain can be viewed as a linear, and hence continuous transformation of the simplex of distributions, which is closed and compact, already Brouwer's fixed point theorem guarantees that the rank always exists. Finding a meaningful, useful notion of rank is another matter.

First of all, as already mentioned, networks often decompose into loosely connected subnets. In the long run, all traffic is likely to get captured in some such subnet. This results in multiple stationary distributions, each concentrated in a closed subnet, zero otherwise. Dynamics derived directly from the network graph therefore result in uninformative ranking data. In order to assure that the relevant Markov chains are irreducible and aperiodic, and thus induce unique and nondegenerate stationary distributions, network dynamics usually need to be perturbed, using a damping and stabilizing factor such as teleportation. Another sort of problems arise when the unique stationary distribution is not an attractor, or when the rate of convergence is unfeasibly slow [7,3].

While very important in concrete applications, these problems, and their solutions, have less impact on the conceptual analyses pursued in this paper. *We shall henceforth assume that all processes have been adjusted to induce unique and effectively computable ranking.*⁴

5.1 Promotion and reputation

We now explain the intuition behind the simplest notions of rank.

In social terms, the push coefficient $A_{ij}^\triangleleft = \frac{A_{ij}}{A_{i\bullet}}$ can be interpreted as measuring how much i supports (or advocates) j . The concept of *promotion* can then be formalized as a probability distribution r^\triangleleft , such that $r_i^\triangleleft = \sum_{k=1}^N A_{ik}^\triangleleft r_k^\triangleleft$. In words, the promotion rank (or *push rank*) r_i^\triangleleft of a node i is the sum of the promotion ranks r_k^\triangleleft of its children nodes, each allocated to i according to the push coefficient A_{ik}^\triangleleft , measuring i 's support for k .

Dually, the pull coefficient A_{ij}^\triangleright can be interpreted as measuring how much i trusts j . The concept of *reputation* can then be formalized as a probability distribution r^\triangleright , such that $r_j^\triangleright = \sum_{k=1}^N r_k^\triangleright A_{kj}^\triangleright$. This reputation rank (or *pull rank*) r_i^\triangleright of a node i is thus the sum of the reputation ranks r_k^\triangleright of its parent nodes, each allocated according to the pull coefficient A_{kj}^\triangleright , of k 's trust in j .

³ The original version allowed A_{ij}^\triangleright to be 0, if $A_{i\bullet}$ is 0, i.e. if i is a "sink-hole", and the teleportation factor was added to save dynamics from such sinkholes. Other modifications were introduced later.

⁴ This implies that all notions of dynamics that we consider have a tacit damping factor. We do not display it only because it needlessly complicates formulas.

Gathering the promotion values in a column vector r^\triangleleft and the reputation values in a row vector r^\triangleright , we can rewrite the definitions of r^\triangleleft and r^\triangleright in the matrix form

$$r^\triangleleft = A^\triangleleft r^\triangleleft \quad r^\triangleright = r^\triangleright A^\triangleright$$

The refined notions of promotion r^\blacktriangleleft and reputation r^\blacktriangleright are defined and interpreted along the same lines, as the stationary distributions of the processes A^\blacktriangleleft and A^\blacktriangleright respectively.

5.2 Expected flow

While dynamics of reputation has been studied for a long time [11,10], and with increased attention recently, since it become a crucial tool of Web search [19,13], the dual dynamics of promotion does not seem to have attracted much attention. We need both notions to define the expected traffic flow.

The expected flow from j to k , under the assumption that they are independent, is caused only by a "traffic pressure", resulting from the pull to j and the push from k . Following this idea, we define

$$r_{jk}^\blacktriangleright = r_j^\blacktriangleright r_k^\blacktriangleleft \quad (1)$$

The expected flow r^\blacktriangleright is thus a probability distribution over $N \times N$, which can be represented as the matrix $r^\blacktriangleright = r^\blacktriangleleft \cdot r^\blacktriangleright$, obtained by multiplying the column vector r^\blacktriangleleft and the row vector r^\blacktriangleright . Since r^\blacktriangleleft and r^\blacktriangleright are the principal eigenvectors of A^\blacktriangleleft and A^\blacktriangleright , r^\blacktriangleright is the unique distribution satisfying $r^\blacktriangleright = A^\blacktriangleleft \cdot r^\blacktriangleright \cdot A^\blacktriangleright$, i.e. $r_{jk}^\blacktriangleright = \sum_{i=1}^N \sum_{\ell=1}^N A_{ij}^\blacktriangleright r_{i\ell}^\blacktriangleright A_{k\ell}^\blacktriangleleft$. Intuitively, this means that the flow pressure from i to ℓ propagates to cause a flow pressure from j to k proportionally to the force of the traffic from i to j and to the force of traffic flows from k to ℓ — *provided* that j and k are independent. In order to measure their dependency, we attempt to capture how the *actual flows* from i to ℓ (rather than mere flow pressure) may get diverted, say by the high costs and the low capacities, to cause actual flows from j to k .

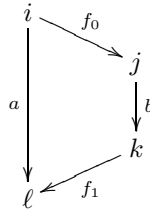
6 Path networks

Definition 2. Given a v -closed network $A = (R \xleftarrow{\gamma} E \xrightarrow[\varrho]{\delta} N)$, we define the path network

$$\hat{A} = (R \xleftarrow{\gamma} \hat{E} \xrightarrow[\varrho]{\delta} \hat{N}), \text{ where } \hat{N} = E, \text{ and } \hat{E} = \sum_{a,b \in E} \hat{E}_{ab}, \text{ with}$$

$$\hat{E}_{ab} = \{f = \langle f_0, f_1 \rangle \in E_{ij} \times E_{k\ell} \mid \gamma(f_0) + \gamma(b) + \gamma(f_1) - \gamma(a) \leq v - 2d\} \quad (2)$$

$$\gamma(f) = 2d + \gamma(f_0) + \gamma(b) + \gamma(f_1) - \gamma(a) \quad (3)$$



Dynamics of path selection. Recalling that $\hat{A}_{ab} = \sum_{f \in \hat{E}_{ab}} 2^{-\gamma(f)}$, we define the forward and the backward dynamics, and the pull rank and the push rank just like before:

$$\begin{aligned} \hat{A}_{ab}^\triangleright &= \frac{\hat{A}_{ab}}{\hat{A}_{a\bullet}} & \hat{A}_{ab}^\triangleleft &= \frac{\hat{A}_{ab}}{\hat{A}_{\bullet b}} \\ \hat{r}_b &= \sum_{a \in \hat{N}} \hat{r}_a \hat{A}_{ab}^\triangleright & \hat{r}_a &= \sum_{b \in \hat{N}} \hat{A}_{ab}^\triangleleft \hat{r}_b \end{aligned}$$

Intuitively, $\hat{A}_{ab}^\triangleright$ is now the probability that traffic through a is diverted to b (rather than to some other path); while $\hat{A}_{ab}^\triangleleft$ is the probability that traffic through b is diverted from a (and not from some other path). The pull rank \hat{r}_b , i.e. the probability that b will be traversed, can thus be understood as its *attraction*; whereas \hat{r}_a^\triangleleft is the probability that a will be avoided.

Using the pull rank of the paths, we can now define the *node attraction* between j and k to be the total attraction of all paths between them:

$$\hat{r}_{jk} = \sum_{j \xrightarrow{b} k} \hat{r}_b \quad (4)$$

The idea is that this notion of attraction the nodes will allow us to refine the estimate of the traffic bias v as described in section 2. In particular, consider *attraction bias*

$$\Upsilon_{jk} = \hat{r}_{jk} - r_{jk}^\blacktriangleright \quad (5)$$

To motivate this, note that expanding the formula for $r_{jk}^\blacktriangleright$ in section 5.2 shows that r^\blacktriangleright is the stationary distribution of the Markov chain $A^\blacktriangleright = \left(A_{(ij)(k\ell)}^\blacktriangleright \right)_{N^2 \times N^2}$, where

$$A_{(ij)(k\ell)}^\blacktriangleright = \frac{A_{ij} A_{j\bullet} A_{\bullet k} A_{k\ell}}{A_{i\bullet} A_{\bullet\bullet}^2 A_{\bullet\ell}} \quad \text{and} \quad r_{jk}^\blacktriangleright = \sum_{i, \ell \in N} A_{(ij)(k\ell)}^\blacktriangleright r_{i\ell}^\blacktriangleright$$

On the other hand, the node attraction \hat{r} turns out to be a stationary distribution of a process that refines A^\blacktriangleright .

Definition 3. Given a network A , its attraction dynamics is a Markov chain $\hat{A} = \left(\hat{A}_{(ij)(k\ell)} \right)_{N^2 \times N^2}$, with the entries

$$\hat{A}_{(ij)(k\ell)} = \frac{A_{ij} A_{jk} A_{k\ell}}{A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}} \quad (6)$$

where $A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell} = \sum_{m,n \in N} A_{im} A_{mn} A_{n\ell}$.

Proposition 1. Suppose that a given network A is v -complete for a sufficiently large v . Then the node attraction \hat{r} , defined in (4), is the stationary distribution of its attraction dynamics (6). In other words, for every j, k holds

$$\hat{r}_{jk} = \sum_{i, \ell \in N} \hat{A}_{(ij)(k\ell)} \hat{r}_{i\ell} \quad (7)$$

The proof is in the Appendix. It is based on the following lemma.

Lemma 1. For a network A , which is v -complete for a sufficiently large cutoff value v , the following equations hold for $i \xrightarrow{a} \ell$ and $j \xrightarrow{b} k$

$$\hat{A}_{ab} = \frac{A_b}{4^d A_a} A_{ij} A_{k\ell} \quad (8)$$

$$\sum_{j \xrightarrow{c} k} \hat{A}_{ac} = \frac{1}{4^d A_a} A_{ij} A_{jk} A_{k\ell} \quad (9)$$

$$\hat{A}_{a\bullet} = \frac{1}{4^d A_a} A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell} \quad (10)$$

On the other hand, proposition 1 implies the following corollary, which establishes that formula (5) can be used to measure the attraction bias, as intended.

Corollary 1. *The directed reputation and promotion ranks are the marginals of the node attraction*

$$\sum_{k \in N} \hat{r}_{jk} = r_j^\blacktriangleright \quad (11)$$

$$\sum_{j \in N} \hat{r}_{jk} = r_k^\blacktriangleleft \quad (12)$$

Interpretation. To understand the meaning of attraction bias, consider a v -complete network A , with the forward-out and backward-in dynamics. The pull rank r_j^\blacktriangleright tells how likely it is that a randomly sampled traffic path arrives to j ; whereas the push rank r_k^\blacktriangleleft tells how likely it is that a randomly sampled traffic path departs from k .

On the other hand, the attraction dynamics in the induced path network \hat{A} gives the node attraction \hat{r}_{jk} , which tells how likely it is that a randomly sampled traffic path traverses a path from j to k . In summary, we have

$$\begin{aligned} r_j^\blacktriangleright &= \text{Prob}(\bullet \xrightarrow{\xi} j \mid \xi \in A^\blacktriangleright) \\ r_k^\blacktriangleleft &= \text{Prob}(k \xrightarrow{\xi} \bullet \mid \xi \in A^\blacktriangleleft) \\ \hat{r}_{jk} &= \text{Prob}(j \xrightarrow{\xi} k \mid \xi \in \hat{A}) \end{aligned}$$

Although the notation suggests that r^\blacktriangleright , r^\blacktriangleleft , and \hat{r} are sampled from different processes, corollary 1 establishes that \hat{r} is in fact the joint distribution of r^\blacktriangleright and r^\blacktriangleleft .

Nevertheless, a diligent reader will surely notice a twist of j and k in the last three equations, and wonder why is the probability that traffic goes from j to k related with the probabilities that it arrives *to* j , and that it departs *from* k ? — The answer to this question makes the forward-*out* and the backward-*in* dynamics into a more interesting example than its many dynamical cousins. Briefly, if the surfers are more likely to flow with $\bullet \rightarrow j$ if the capacity of the links out of j is higher, and if they are more likely to flow with $k \rightarrow \bullet$ if the capacity of the links into k is higher, then the surfers are most likely to follow both these flows, i.e. into j and out of k — if there is a high capacity of the links $j \rightarrow k$.

Mutual information of the inputs and the outputs. The fact that \hat{r} is the joint distribution of the processes expressed by r^\blacktriangleright and r^\blacktriangleleft allows us to extract their *mutual information* [4]

$$I(r^\blacktriangleright ; r^\blacktriangleleft) = D(\hat{r} \parallel r^\blacktriangleright r^\blacktriangleleft) = \sum_{j=1}^N \sum_{k=1}^N \hat{r}_{jk} \log \frac{\hat{r}_{jk}}{r_j^\blacktriangleright r_k^\blacktriangleleft}$$

Its expression in terms of relative entropy $D(\hat{r} \parallel r^\blacktriangleright r^\blacktriangleleft)$ [*ibidem*] shows that it measures how much we lose, in the efficiency of encoding of \hat{r} if we assume that r^\blacktriangleright and r^\blacktriangleleft are mutually independent. Intuitively, the mutual information $I(r^\blacktriangleright ; r^\blacktriangleleft)$ can thus be taken as a measure of the *locality* of information processing in A . If this is an entirely local process, then every path must begin and end at the same node, and the random walks δ and ϱ , selecting the sources and the destinations of the paths, must coincide. But if $\delta = \varrho$, then the push rank and the pull rank must obey the same distribution $r^\blacktriangleleft = r^\blacktriangleright = r$, and their mutual information is $I(r^\blacktriangleright ; r^\blacktriangleleft) = H(r)$, their entropy. In the other extreme case, the random walks δ and ϱ are independent⁵, and their joint distribution is just the product of their distributions $\hat{r}_{jk} = r_j^\blacktriangleright r_k^\blacktriangleleft$. Their mutual information is then $I(r^\blacktriangleright ; r^\blacktriangleleft) = 0$.

7 Conclusions and future work

When the Web is viewed as a global data store, the problem of its semantics is the problem of determining a uniform meaning for the data published by its various participants. The search engines are dealing with

⁵ The theorem in the appendix suggests that they are similarly distributed, up to a scale factor.

this problem on the level of the human-Web interaction (e.g., distinguishing the meanings of the word "jaguar", sometimes denoting a car, sometimes an animal [12], or deciding whether "Paris Hilton", in a given context, refers to a person or to a hotel, etc.), whereas the Semantic Web project [1] deals with the computer-Web interactions. When the Web is viewed as a computer, the problem of its semantics is not just a matter of assigning some meanings to some data stored in it, but also to its data processing operations. For programming languages, this is what we usually call operational semantics. However, unlike a programming language, the Web, and other spontaneously evolving networks, do not have a formally defined set of data structures and operations: data are transformed by many random walks, running concurrently. Operational semantics of network computation requires a toolkit for incremental analysis of such processes. In this paper, we described a path ranking method, which is may become a useful piece of that toolkit. Now we sketch a way to test it experimentally. Using the notion of attraction bias, we lift the graph theoretic notion of (maximal) *clique* into rank analysis, while retaining network dynamics as a graph structure over such generalized cliques. We call these generalized cliques *concepts* and the links between them *associations*.

Communities and concepts. Taking the notion of attraction bias back to the idea of communities as sets of nodes with high cohesion, from which we started in the Introduction, we now reformulate the notion of cohesion in a different norm (ℓ_∞ instead of ℓ_1), and define cohesion of a set of nodes $U \subseteq N$ to be their minimal symmetric attraction bias

$$\mathcal{Y}(U) = \bigwedge_{i,j \in U} (\mathcal{Y}_{ij} \vee \mathcal{Y}_{ji})$$

For each $\varepsilon \in [0, 1]$, we define an ε -community to be a set of nodes $U \subseteq N$ such that $\mathcal{Y}(U) \geq \varepsilon$. Denoting by $\wp_\varepsilon N$ the set of ε -communities, note that $\varepsilon_1 \leq \varepsilon_2$ implies that $\wp_{\varepsilon_1} N \supseteq \wp_{\varepsilon_2} N$. The partial ordering of $U, V \in \wp_\varepsilon N$ is given by $U \sqsubseteq V \iff U \subseteq V \wedge \mathcal{Y}(U) \leq \mathcal{Y}(V)$. This gives a *directed complete partial order* (*dcpo*). It is not a lattice because some communities cannot be extended by new nodes without decreasing their cohesion; so there are pairs of communities that cannot be joined, and do not have an upper bound. However, *directed* sets of communities (i.e., where each pair has an upper bound) do have least upper bounds, which are just their set theoretic unions. Directed complete partial orders are often used in denotational semantics of programming languages [5]. According to that interpretation, communities can be thought of as pieces of *partial information*, their \sqsubseteq -ordering as the increase of information, and the existence of an upper bound of two communities as the *consistency* of the informations that they carry.

The maximal elements of $\wp_\varepsilon N$, i.e. the communities that cannot be extended by new nodes without losing cohesion, can be construed as ε -concepts. A set $U \in \wp_\varepsilon N$ is thus an ε -concept if $\mathcal{Y}(\{i, j\}) \geq \varepsilon$ holds for all $i, j \in U$, but for every $k \in N \setminus U$ there is a $j \in U$ such that $\mathcal{Y}(\{k, j\}) < \varepsilon$.

The community and concept structure of a network A can be analyzed by studying the sequence of hypergraphs A_ε , where the ε -concepts, or the ε -communities approximating them, are viewed as hyperedges. The sequence $(A_\varepsilon)_{\varepsilon \in [0, 1]}$ decreases as the cohesion parameter ε increases, and the highly cohesive communities and concepts can be feasibly analyzed.

A level further, concepts and communities can be viewed as the nodes of a network. The most interesting definition of the links between them, intuitively thought of as associations, is based on a variant of a path network, complementing definition 2. A sketch of this definition is in the next, final subsection.

Associations. Let \mathcal{N}^ε denote the set of ε -concepts in a network A .

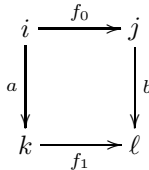
The *concept network* \mathcal{A}^ε , induced by a network A , has the ε -concepts as its nodes. Its edges are called *concept associations*. The set of associations between $U, V \in \mathcal{N}^\varepsilon$ is

$$\mathcal{E}_{UV}^\varepsilon = \sum_{U \xrightarrow{a} U \cap V} \sum_{U \cap V \xrightarrow{b} V} \tilde{E}_{ab} \quad (13)$$

where $U \xrightarrow{\xi} V$ abbreviates $\delta(\xi) \in U \wedge \varrho(\xi) \in V$, and

$$\tilde{E}_{ab} = \{f = \langle f_0, f_1 \rangle \in E_{ij} \times E_{kl} \mid \gamma(f_0) + \gamma(b) \leq v - d \text{ and } \gamma(a) + \gamma(f_1) \leq v - d\}$$

An association $f \in \mathcal{A}_{UV}$ is thus a quadruple $f = \langle a, b, f_0, f_1 \rangle$



such that $i, j, k \in U$ and $j, k, \ell \in V$. Its cost is $\gamma(f) = \gamma(f_0) + \gamma(b) - \gamma(a) - \gamma(f_1)$. The cost of an association from U to V is lower if the traffic from $i \in U$ to $\ell \in V$ gets less costly when it crosses to V earlier.

While the general network analysis tools apply to concept networks, the various notions of dynamics acquire new meanings on this level. At this point, understanding which of the possible interpretations may lead to useful tools for extracting and analyzing the relevant concepts, processed in a network, seems to call for experimentation with real data.

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Appendix: Proofs

Proof (of lemma 1(8)). The first claim is that there is a sufficiently large v such that $\gamma(f_0) + \gamma(b) + \gamma(f_1) \leq v - 2d$ holds for all $f_0 \in E_{ij}$ and $f_1 \in E_{k\ell}$. Since b and d are fixed, the claim is clear if E_{ij} and $E_{k\ell}$ are finite. Since A is assumed to be truncated complete, an infinite set of paths can only be generated from the links with a cost ≤ 0 . So the costs of the elements of E_{ij} and $E_{k\ell}$ are in any case bounded.

But if all $f_0 \in E_{ij}$ and $f_1 \in E_{k\ell}$ satisfy $\gamma(f_0) + \gamma(b) + \gamma(f_1) \leq v - 2d$, then $\hat{E}_{ab} = E_{ij} \times E_{k\ell}$. Unfolding the definition of \hat{A}_{ab} and using (3) we get

$$\begin{aligned}\hat{A}_{ab} &= \sum_{f \in \hat{E}_{ab}} 2^{-\gamma(f)} \\ &= \sum_{f_0 \in E_{ij}} \sum_{f_1 \in E_{k\ell}} 2^{-2d - \gamma(f_0) - \gamma(b) - \gamma(f_1) + \gamma(a)} \\ &= 4^{-d} \frac{2^{-\gamma(b)}}{2^{-\gamma(a)}} \sum_{f_0 \in E_{ij}} 2^{-\gamma(f_0)} \sum_{f_1 \in E_{k\ell}} 2^{-\gamma(f_1)} \\ &= \frac{A_b}{4^d A_a} A_{ij} A_{k\ell}\end{aligned}$$

1(9) follows directly from 1(8), unpacking $A_e = 2^{-\gamma(e)}$. And 1(10) then follows from 1(9):

$$\begin{aligned}\hat{A}_{a\bullet} &= \sum_b \hat{A}_{ab} \\ &= \sum_{j,k \in N} \sum_{j \xrightarrow{b} k} \hat{A}_{ab} \\ &= \sum_{j,k \in N} \frac{1}{4^d A_a} A_{ij} A_{jk} A_{k\ell} \\ &= \frac{1}{4^d A_a} A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}\end{aligned}$$

Proof (of proposition 1(7)). We unfold the definition of \hat{r} and then use (9) and (10):

$$\begin{aligned}\hat{r}_{jk} &= \sum_{j \xrightarrow{b} k} \hat{r}_b \\ &= \sum_{j \xrightarrow{b} k} \sum_a \frac{\hat{A}_{ab}}{\hat{A}_{a\bullet}} \hat{r}_a \\ &= \sum_a \frac{\sum_{j \xrightarrow{b} k} \hat{A}_{ab}}{\hat{A}_{a\bullet}} \hat{r}_a \\ &= \sum_a \frac{\frac{1}{4^d A_a} A_{ij} A_{jk} A_{k\ell}}{\frac{1}{4^d A_a} A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}} \hat{r}_a \\ &= \sum_{i,\ell \in N} \sum_{i \xrightarrow{a} \ell} \frac{A_{ij} A_{jk} A_{k\ell}}{A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}} \hat{r}_a \\ &= \sum_{i,\ell \in N} \frac{A_{ij} A_{jk} A_{k\ell}}{A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}} \sum_{i \xrightarrow{a} \ell} \hat{r}_a \\ &= \sum_{i,\ell \in N} \hat{A}_{(ij)(k\ell)} \hat{r}_{i\ell}\end{aligned}$$

Proof (of corollary 1(11)). We set $q_j = \sum_{k \in N} \hat{r}_{jk}$ and expand \hat{r}_{jk} using proposition 1(7), to show that q is the stationary distribution of the process A^\triangleright :

$$\begin{aligned}
q_j &= \sum_{k \in N} \hat{r}_{jk} \\
&= \sum_{k, i, \ell \in N} \hat{A}_{(ij)(k\ell)} \hat{r}_{i\ell} \\
&= \sum_{k, i, \ell \in N} \frac{A_{ij} A_{jk} A_{k\ell}}{A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}} \hat{r}_{i\ell} \\
&= \sum_{i, \ell \in N} \frac{A_{ij} A_{j\bullet}}{A_{i\bullet} A_{\bullet\bullet}} \hat{r}_{i\ell} \\
&= \sum_{i \in N} A_{ij}^\triangleright \frac{A_{j\bullet}}{A_{\bullet\bullet}} \sum_{\ell \in N} \hat{r}_{i\ell} \\
&= \sum_{i \in N} A_{ij}^\triangleright q_i
\end{aligned}$$

Since r^\triangleright is by definition the stationary point of A^\triangleright , the uniqueness implies $q = r^\triangleright$.

To prove 1(12), we set $q_k = \sum_{j \in N} \hat{r}_{jk}$ and proceed similarly:

$$\begin{aligned}
q_k &= \sum_{j \in N} \hat{r}_{jk} \\
&= \sum_{j, i, \ell \in N} \hat{A}_{(ij)(k\ell)} \hat{r}_{i\ell} \\
&= \sum_{j, i, \ell \in N} \frac{A_{ij} A_{jk} A_{k\ell}}{A_{i\bullet} A_{\bullet\bullet} A_{\bullet\ell}} \hat{r}_{i\ell} \\
&= \sum_{i, \ell \in N} \frac{A_{\bullet k} A_{k\ell}}{A_{\bullet\bullet} A_{\bullet\ell}} \hat{r}_{i\ell} \\
&= \sum_{\ell \in N} \frac{A_{\bullet k}}{A_{\bullet\bullet}} A_{k\ell}^\triangleleft \sum_{i \in N} \hat{r}_{i\ell} \\
&= \sum_{\ell \in N} A_{k\ell}^\triangleleft q_\ell
\end{aligned}$$

Since r^\triangleleft is by definition the stationary point of A^\triangleleft , the uniqueness implies $q = r^\triangleleft$.